# Quiz 2 solutions

1. For the following pairs of groups $(H, K)$, describe all non-trivial semi-direct products of the form $H \ltimes_\psi K$ and $K \ltimes_{\psi'} H$.

   (a) $(H, K) = (\mathbb{Z}, \mathbb{Z}_2)$.

   (b) $(H, K) = (\mathbb{Z}_4, \mathbb{Z}_8)$.

   **Solution.** (a) There exists no non-trivial semi-direct of the form $\mathbb{Z} \ltimes_\psi \mathbb{Z}_2$ as such a semi-direct product is determined by a non-trivial homomorphism $\psi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_2)$, which does not exist since $\mathrm{Aut}(\mathbb{Z}_2)$ is trivial. Now, since $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, there exists only one non-trivial homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$, which is the identity homomorphism $\psi : \mathbb{Z}_2 \to \mathbb{Z}_2$. Hence, $\psi(= id)$ determines a non-trivial semi-direct product $\mathbb{Z}_2 \ltimes_\psi \mathbb{Z}$. (Describe the group operation on $\mathbb{Z}_2 \ltimes_\psi \mathbb{Z}$.)

   (b) We know from 5.2 (vi)(b) of the Lesson Plan that any non-trivial semi-direct product of the form $\mathbb{Z}_m \ltimes_\psi \mathbb{Z}_n$ is determined by a (non-trivial) $k \in U_n$ satisfying $k^m \equiv 1 \pmod{n}$. Since $U_4 = \{1, 3\}$ and $U_8 = \{1, 3, 5, 7\}$, there exists four such non-trivial semi-direct products, namely:

   (i) $\mathbb{Z}_8 \ltimes_3 \mathbb{Z}_4$,

   (ii) $\mathbb{Z}_4 \ltimes_3 \mathbb{Z}_8$,

   (iii) $\mathbb{Z}_4 \ltimes_5 \mathbb{Z}_8$, and

   (iv) $\mathbb{Z}_4 \ltimes_7 \mathbb{Z}_8$.

2. Let $G$ be a group of order $pq$, where $p$ and $q$ are distinct primes with $p > q$. Then show that $G \cong \mathbb{Z}_q \ltimes_\psi \mathbb{Z}_p$.

   **Solution.** By the First Sylow Theorem, $G$ has Sylow $p$- subgroup $P$ or order $p$, and a Sylow $q$-subgroup $Q$ of order $q$. Furthermore, since $n_p \equiv 1 \pmod{p}$ and $n_p \mid q$, it follows that that $n_p = 1$. Thus, $P$ is unique and by 4.4 (x) of the Lesson Plan, it follows that $P \lhd G$.

   Now consider the action $Q \curvearrowright^c P$ by conjugation (see Lesson Plan 4.3.3.), which is well-defined since $P \lhd Q$. For a fixed $g \in Q$, the map $\varphi_g : P \to P, h \mapsto ghg^{-1}$ defines an automorphism of $P$, that is, $\varphi_g \in \mathrm{Aut}(P)$, for each $g \in Q$. Thus, the permutation representation

$$\Psi_{Q \curvearrowright^c P} : Q \to S(P) : g \xmapsto{\Psi_{Q \curvearrowright^c P}} \varphi_g$$

of the action $Q \curvearrowright^c P$ defines a homomorphism from $Q \to \text{Aut}(P)$. Therefore, taking $\Psi' = \Psi_{Q \curvearrowright^c P}$, we see that $Q \ltimes_{\Psi'} P$ is a well-defined a semi-direct product. Furthermore, since $P \cong \mathbb{Z}_p$ and $Q \cong \mathbb{Z}_q$, we see that $Q \ltimes_{\psi'} P \cong \mathbb{Z}_q \ltimes_{\psi} \mathbb{Z}_p$, where $\Psi = \Psi_{\mathbb{Z}_q \curvearrowright^c \mathbb{Z}_p}$ is the corresponding permutation representation associated with the conjugation action $\mathbb{Z}_q \curvearrowright^c \mathbb{Z}_p$. (Verify this!)

Now, by the assertion in 2.3 (vi) of the Lesson Plan, we see that the internal direct product $QP \leq G$ and by 2.2 (xiii) of the Lesson Plan, we have that $|QP| = qp$. Hence, it follows that $G = QP$. Finally, our assertion follows from the fact that the map $Q \ltimes_{\psi'} P \to QP : (q, p) \to qp$ defines an isomorphism. (Verify this!)